Hydromagnetic Instability Conditions for Viscoelastic Non-Newtonian Fluids

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The effect of a horizontal magnetic field and a non-Newtonian stress tensor, as described by the Walters B′ model, on the instability of two second order fluids of high kinematic viscosities and viscoelasticities is investigated. For the potentially stable configuration, it is found that the system is stable or unstable for a wavenumber range depending on the kinematic viscoelasticity. For the potentially unstable configuration, it is found that the stability criterion is dependent on orientation and magnitude of the magnetic field which is found to stabilize a certain range of the unstable configuration related to the viscoelasticity values. The behaviour of growth rates with respect to Alfvén velocities are examined analytically, and it is found that the magnetic field has a dual role on the instability problem. For the exponentially varying stratifications, the system is found to be stable or unstable for the stable and unstable stratifications under certain physical conditions, and the growth rates are found to increase or decrease with increasing the stratification parameter values, according to some restrictions satisfied by the chosen wavenumbers range.

Key words: Hydrodynamic Stability; Non-Newtonian Fluid Flows; Magnetohydrodynamics.

1. Introduction

The instability of the plane interface separating two fluids when one is accelerated towards the other, or when one is superposed over the other has been studied by several authors. Chandrasekhar [1] has given a detailed account of these investigations. Roberts [2] extended the analysis to the case of equal kinematic viscosities, while Gerwin [3] has studied the case of compressible streaming fluids. Kruskal and Schwarzschild [4] have considered the stability of an inviscid plasma of infinite conductivity supported against gravity by a horizontal magnetic field. Hide [5] studied the case of a conducting fluid with a transverse magnetic field, and he found that the magnetic field considerably stabilizes the configuration. The stability of the plane interface separating two incompressible superposed viscous fluids acted upon by a uniform magnetic field has been studied by Bhatia and Steiner [6]. A good account of hydrodynamic stability problems has been given by Drazin and Reid [7], Joseph [8], and El-Sayed and others [9–11]. The fluids have been considered to be Newtonian in all the above mentioned studies.

Most liquids exhibit viscous and elastic properties for sufficiently small time scales. Studies on light scattering and molecular dynamic simulations show that in order to recover experimental results, one must generalize Newton's law relating linearly the extra stress tensor and the rate of strain [12]. In particular, this generalization is necessary in polymeric fluids, glass forming systems, etc. Moreover, in these systems the stress tensor enters as independent variab le in the hydrodynamic description [13]. The dynamics of the stress tensor give rise to a rich variety of hydrodynamic effects. Molten plastics, petroleum oil additives, and whipped cream are examples of incompressible viscoelastic fluids.

The stability of a layer of viscoelastic (Oldroyd) fluid heated from below and subject to a magnetic field has been studied by Sharma [14]. He has also studied the thermal instability of a layer of Oldroydian viscoelastic fluid acted on by a uniform rotation, and he found that the rotation has a stabilizing as well as destabilizing effect under certain conditions. This is in contrast to the thermal instability of a Maxwellian viscoelastic fluid in the presence of rotation studied by Bhatia [6], where the rotation has a destabilizing effect. Also Sharma and Sharma [15] have studied the stability of the plane interface separating two viscoelastic (Oldroyd) superposed fluids of uniform densities. Fredricksen [16] has given a good review of non-Newtonian fluids, whereas excellent reviews about the subject of the stability of viscoelastic fluids have been given by Joseph [8], and Martinez-Mardones

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and Pérez-Garcia [17]. On the other hand, long wavelength analyses of two-layer Couette flows have been carried out by Yih [18] for Newtonian fluids, and by Li [19], Walters and Keeley [20], and Chen [21] for Oldroyd-B, or upper convected Maxwell fluids; also short wavelength analyses for such fluids have been put forth by Renardy [22]. Instabilities occur because of stratification of fluid properties. Although surface tension can be counted on to stabilize short waves, long waves can be easily become unstable. For a Newtonian or a viscoelastic fluid, density stratification with the heavy fluid on top leads to long-wave instabilities. A viscosity stratification produces an instability if the Reynolds number exceeds zero [18], although, if the thin layer is the less viscous, stability is maintained at low Reynolds numbers [22, 23].

There are many non-Newtonian (or viscoelastic) fluids that cannot be characterized by Maxwell's or Oldroyd's constitutive equations. One of such class of viscoelastic fluids is Walters B′ liquid. Chakraborty and Sengupta [24] have studied the flow of an unsteady viscoelastic Waters B′ conducting fluid through two porous concentric non-conducting infinite circular cylinders rotating with different angular velocities in the presence of a uniform axial magnetic field. In other studies, Sharma and Kumar [25] have studied the steady flow and heat transfer of Walters B′ fluid through a porous pipe of uniform circular cross-section. They also studied the unsteady flow of a Walters fluid (model B′) down an open inclined channel under gravity. For excellent works about the subject, see the recent papers of Sharma and his colaborators [26, 27].

In this paper we have investigated the stability of the plane interface separating two superposed incompressible viscoelastic Walters B′ fluids of equal kinematic viscosities and viscoelasticities and different densities, pervaded by a uniform horizontal magnetic field in addition to a constant gravity field. The case of exponentially varying stratifications in fluid density, velocity, viscoelasticity, and magnetic field has been also investigated, and the stability conditions are obtained and discussed.

2. Basic and Perturbation Equations

We consider a static state in which an incompressible Walters B′ viscoelastic fluid of variable density prevaded by a uniform magnetic field $\boldsymbol{H}(H_x, H_y, 0)$ is arranged in horizontal strata, under the action of gravity $\boldsymbol{g}\,(0, 0, -g)$. The character of the equilibrium of this initial static state is determined by supposing that the system is slightly disturbed and then following its further evolution.

Let ϱ, p, and $\boldsymbol{u}\,(0, 0, 0)$ denote, respectively, the density, the pressure, and the velocity of the considered fluid. Then the momentum balance and mass balance quations for Walters B′ incompressible viscoelastic fluid are

$$\rho\left[\frac{\partial \mathrm{u}}{\partial t}+(\mathrm{u}\cdot\nabla)\,\mathrm{u}\right]=-\nabla p+\rho\gamma+\frac{\mu_e}{4\pi}(\nabla\times\mathrm{H})\times\mathrm{H}$$
$$+\rho\left(\nu-\nu'\frac{\partial}{\partial t}\right)\nabla^2\mathrm{u}+\left(\frac{\mathrm{d}\mu}{\mathrm{d}z}-\frac{\partial}{\partial t}\frac{\mathrm{d}\mu'}{\mathrm{d}z}\right)\left(\frac{\partial\omega}{\partial \mathrm{x}}+\frac{\partial \mathrm{u}}{\partial z}\right) \tag{1}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{u}=0, \tag{2}$$

$$\frac{\partial \boldsymbol{H}}{\partial t}=\boldsymbol{\nabla}\times(\boldsymbol{u}\times\boldsymbol{H}), \tag{3}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{H}=0, \tag{4}$$

where μ, μ', $\nu(=\mu/\varrho)$, and $\nu'(=\mu'/\varrho)$ stand for the viscosity, viscoelasticity, kinematic viscosity, and kinematic viscoelasticity, respectively, μ_e is the magnetic permeability which is assumed to be constant, and $\boldsymbol{x}=(x, y, z)$.

Since the density of a fluid particles remains unchanged as we follow it with its motion, we have

$$\frac{\partial\rho}{\partial t}+(\boldsymbol{u}\cdot\boldsymbol{\nabla})\,\rho=0. \tag{5}$$

This additional equation needs to be satisfied as the fluid is heterogeneous.

Let $\delta\boldsymbol{u}\,(u, v, w)$, $\delta\varrho$, δp, and $\boldsymbol{h}\,(h_x, h_y, h_z)$ denote, respectively, the perturbations in fluid velocity $\boldsymbol{u}\,(0, 0, 0)$, density ρ, pressure p, and the magnetic field $\boldsymbol{H}\,(H_x, H_y, 0)$. Then the linearized perturbation equations appropriate to the problem are

$$\varrho\frac{\partial\delta\boldsymbol{u}}{\partial t}=-\boldsymbol{\nabla}\delta p+\boldsymbol{g}\,\delta g+\frac{\mu_e}{4\pi}(\boldsymbol{\nabla}\times\boldsymbol{h})\times\boldsymbol{H}$$
$$+\varrho\left(\nu-\nu'\frac{\partial}{\partial t}\right)\boldsymbol{\nabla}^2\delta\boldsymbol{u}$$
$$+\left(\frac{\mathrm{d}\mu}{\mathrm{d}z}-\frac{\partial}{\partial t}\frac{\mathrm{d}\mu'}{\mathrm{d}z}\right)\left(\frac{\partial\omega}{\partial \boldsymbol{x}}+\frac{\partial\delta\boldsymbol{u}}{\partial z}\right), \tag{6}$$

$$\boldsymbol{\nabla}\cdot\delta\boldsymbol{u}=0, \tag{7}$$

$$\frac{\partial\boldsymbol{h}}{\partial t}=\boldsymbol{\nabla}\times(\delta\boldsymbol{u}\times\boldsymbol{H}), \tag{8}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{h}=0, \tag{9}$$

$$\frac{\partial\delta\varrho}{\partial t}=-\,\omega\frac{\mathrm{d}\varrho}{\mathrm{d}z}. \tag{10}$$

Analyzing the disturbances into normal modes, we assume that the perturbation quantities have a space and a time dependence of the form

$$f(z)\exp(i k_1 x + i k_2 y + n t), \tag{11}$$

where n is the growth rate at which the system departs from the equilibrium, k_1 and k_2 are the horizontal components of the wavenumber, $k^2 = k_1^2 + k_2^2$, and $f(z)$ is a function of z. Using (11), (6)–(10) give

$$\begin{aligned}\varrho n u = &- i k_1 \delta p - \frac{\mu_e H_y}{4\pi}(i k_1 h_y - i k_2 h_x) \\ &+ \varrho(\nu - \nu' n)(D^2 - k^2) u \\ &+ (D\mu - n D\mu')(i k_1 \omega + D u),\end{aligned} \tag{12}$$

$$\begin{aligned}\varrho n v = &- i k_2 \delta p + \frac{\mu_e H_y}{4\pi}(i k_1 h_y - i k_2 h_x) \\ &+ \varrho(\nu - \nu' n)(D^2 - k^2) v \\ &+ (D\mu - n D\mu')(i k_2 \omega + D v),\end{aligned} \tag{13}$$

$$\begin{aligned}\varrho n w = &- D\delta p - g\delta p + \frac{\mu_e H_y}{4\pi}(i k_2 h_z - D h_y) \\ &- \frac{\mu_e H_x}{4\pi}(D h_x - i k_1 h_z) \\ &+ \varrho(\nu - \nu' n)(D^2 - k^2) w \\ &+ (D\mu - n D\mu')(i k_1 \omega + D u),\end{aligned} \tag{14}$$

$$i k_1 u + i k_2 v = -D w, \tag{15}$$

$$n \boldsymbol{h} = i(\boldsymbol{k} \cdot \boldsymbol{H})\,\delta \boldsymbol{u}, \tag{16}$$

$$i k_1 h_x + i k_2 h_y = -D h_z, \tag{17}$$

$$n\,\delta\varrho + w D\varrho = 0 \quad \text{where} \quad D = \mathrm{d}/\mathrm{d}z. \tag{18}$$

Eliminating δp in (12)–(14), and using (15)–(18), we obtain

$$\begin{aligned} n[D(\varrho D w) - k^2 \varrho w] &+ \frac{g k^2}{n} w (D\varrho) \\ &+ \frac{\mu_e (\boldsymbol{k}\cdot\boldsymbol{H})^2}{4\pi n}(D^2 - k^2) w \\ &- \{D[\varrho(\nu - \nu' n)(D^2 - k^2) D w] \\ &- k^2 \varrho(\nu - \nu' n)(D^2 - k^2) w\} \\ &- \{D[(D\mu - n D\mu')(D^2 + k^2) w] \\ &- 2k^2 (D\mu - n D\mu') D w\} = 0. \end{aligned} \tag{19}$$

In the absence of viscoelasticity, i.e. when $\nu' = 0$, (19) reduces to the equation given by Chandrasekhar [1], while in absence of the magnetic field, i.e. when $\boldsymbol{H} = 0$, it reduces to the equation obtained earlier by Sharma and Kumar [26], and their results are therefore recovered.

3. Two Uniform Viscoelastic Fluids

Consider the case of two uniform superposed Walters B′ viscoelastic fluids of densities ϱ_1 and ϱ_2, viscosities μ_1 and μ_2, viscoelasticities μ_1' and μ_2', and magnetic fields $\boldsymbol{H}_1$ and $\boldsymbol{H}_2$, separated by a horizontal boundary at $z = 0$. The subscripts 1 and 2 refer to the lower and the upper fluids, respectively. In each of the two regions of constant ϱ, μ, μ', and $\boldsymbol{H}$, (19) reduces to

$$(D^2 - k^2)(D^2 - K_j^2) w = 0, \tag{20}$$

where

$$K_j^2 = k^2 + \frac{n}{(\nu_j - \nu_j' n)}\left\{1 + \frac{\mu_e (\boldsymbol{k}\cdot\boldsymbol{H}_j)^2}{4\pi\varrho_j n^2}\right\}, \; j = 1, 2.$$

Since w must vanish both when $z \to -\infty$ (in the lower fluid), and $z \to \infty$ (in the upper fluid), the general solution of (20), in the two regions, can be written as

$$w_j = A_j \exp(\pm k z) + B_j \exp(\pm K_j z), \; j = 1, 2, \tag{21}$$

where A_1, B_1, A_2, and B_2 are constants of integration. For the sake of simplicity, we may consider that the Alfvén velocity vectors of the two fluids are the same, so that

$$\boldsymbol{V}_A = \sqrt{\frac{\mu_e}{4\pi\varrho_1}}\,\boldsymbol{H}_1 = \sqrt{\frac{\mu_e}{4\pi\varrho_2}}\,\boldsymbol{H}_2.$$

Thus, the expressions for K_1 and K_2 take the form

$$K_j = \sqrt{k^2 + \frac{n}{(\nu_j - \nu_j' n)}\left\{1 + \frac{(\boldsymbol{k}\cdot\boldsymbol{V}_A)^2}{n^2}\right\}}. \tag{22}$$

The solutions (21) must satisfy certain boundary conditions. The boundary conditions to be satisfied at the interface $z = 0$ are [26] that

$$w, \; D w, \; \text{and} \; (\mu - \mu' n)(D^2 + k^2) w, \tag{23}$$

must be continuous across the interface between the two fluids.

Integrating (19) across the interface $z = 0$, we obtain

$$\begin{aligned}&\left[1 + \frac{(\boldsymbol{k}\cdot\boldsymbol{V}_A)^2}{n^2}\right]\Delta_0(\varrho D w) - \frac{1}{n}\Delta_0[(\mu - \mu' n) \\ &\cdot (D^2 - k^2) D w] = -\frac{g k^2}{n^2}\Delta_0(\varrho) w_0 \\ &-\frac{2k^2}{n}\Delta_0(\mu - \mu' n) D w_0 \quad \text{at } z = 0,\end{aligned} \tag{24}$$

where Δ_0 denotes the jump a quantity experiences at the interface $z = 0$, and w_0 and $D w_0$ are the common values of w_1, w_2, and $D w_1$, $D w_2$, respectively, at $z = 0$.

Applying the boundary conditions (23) and (24) to the solutions given by (21), we obtain

$$A_1+B_1=A_2+B_2\,, \qquad (25)$$

$$k\,A_1+K_1\,B_1=-k\,A_2-K_2\,B_2\,, \qquad (26)$$

$$(\mu_1-\mu_1'\,n)\,[2\,k^2\,A_1+(K_1^2+k^2)\,B_1] = (\mu_2-\mu_2'\,n)\,[2\,k^2\,A_2+(K_2^2+k^2)\,B_2]\,, \qquad (27)$$

and

$$\left[-\alpha_1\,(1+S)+\frac{R}{2}+C\right]A_1+\left[\frac{R}{2}+\frac{K_1}{k}\,C\right]B_1 + \left[-\alpha_2\,(1+S)+\frac{R}{2}-C\right]A_2+\left[\frac{R}{2}+\frac{K_2}{k}\,C\right]B_2=0\,, \qquad (28)$$

where $\alpha_j=\varrho_j/(\varrho_1+\varrho_2), j=1,2$, and

$$R=\frac{g\,k}{n^2}\,(\alpha_2-\alpha_1),\quad S=\frac{(\boldsymbol{k}\cdot\boldsymbol{V}_A)^2}{n^2}\,,$$

$$C=\frac{k^2}{n}\,[\alpha_2\,(\nu_2-\nu_2'\,n)-\alpha_1\,(\nu_1-\nu_1'\,n)]\,.$$

Equations (25)–(28) can be written, in matrix notation, in the form of a single matrix equation

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14}\\ a_{21} & a_{22} & a_{23} & a_{24}\\ a_{31} & a_{32} & a_{33} & a_{34}\\ a_{41} & a_{42} & a_{43} & a_{44}\end{pmatrix}\begin{pmatrix}A_1\\ B_1\\ A_2\\ B_2\end{pmatrix}=0\,, \qquad (29)$$

where

$$\begin{aligned} &a_{11}=a_{12}=1,\ a_{13}=a_{14}=-1,\\ &a_{21}=a_{23}=k,\ a_{22}=K_1,\ a_{24}=K_2,\\ &a_{31}=2\,k^2\,(\mu_1-\mu_1'\,n),\ a_{33}=-2\,k^2\,(\mu_2-\mu_2'\,n),\\ &a_{32}=(\mu_1-\mu_1'\,n)\,(K_1^2+k^2),\\ &a_{34}=-(\mu_2-\mu_2'\,n)\,(K_2^2+k^2),\\ &a_{41}=(R/2)+C-\alpha_1\,(1+S),\\ &a_{43}=(R/2)-C-\alpha_2(1+S),\\ &a_{42}=(R/2)+(C\ K_1/k),\ a_{44}=(R/2)-(C\ K_2/k)\,. \end{aligned} \qquad (30)$$

The determinant of the linear system of equations represented by (29) must vanishes. The determinant can be reduced by subtracting the first column from the second, the third column from the fourth, and adding the first column to the third. By this procedure we obtain, by expanding the resulting determinant, the dispersion relation

$$\begin{aligned} &(K_1-k)\,\{-2\,C\,n\,[-C\,(K_2/k-1)+\alpha_2(1+S)]\\ &+(\alpha_2\,\nu_2-n\,\alpha_2\,\nu_2')\,(K_2^2-k^2)\,[R-(1+S)]\}\\ &-2\,k\,\{(\alpha_1\,\nu_1-n\,\alpha_1\,\nu_1')\,(K_1^2-k^2)\,[-C\,(K_2/k-1)\\ &+\alpha_2(1+S)]+(\alpha_2\,\nu_2-n\,\alpha_2\,\nu_2')\,(K_2^2-k^2)\\ &\cdot[C\,(K_1/k-1)+\alpha_1\,(1+S)]\}\\ &+(K_2-k)\,\{2\,C\,n\,[C\,(K_1/k-1)+\alpha_1\,(1+S)]\\ &+(\alpha_1\,\nu_1-n\,\alpha_1\,\nu_1')\,(K_1^2-k^2)\,[R-(1+S)]\}=0\,. \end{aligned} \qquad (31)$$

The dispersion equation (31) is quite complicated, as the values of K_1 and K_2 involve square roots. For mathematical simplicity, we make the assumptions that the kinematic viscosities and the kinematic viscoelasticities of the two fluids are the same, i.e. $\nu_1=\nu_2=\nu$, and $\nu_1'=\nu_2'=\nu'$, and the two fluids are of high viscosity and high viscoelasticity. Under these assumptions we have

$$K_j=k+\frac{n\,(1+S)}{2\,k\,(\nu-\nu'\,n)}\,,\quad j=1,2\,. \qquad (32)$$

Substituting the values of $(K_j-k), j=1,2$, from (32) in the dispersion relation (31), and simplifying it, after a little algebra we obtain

$$n^2\,(1-2\,\nu'k^2)+2\,\nu\,k^2\,n + [(\boldsymbol{k}\cdot\boldsymbol{V}_A)^2-g\,k\,(\alpha_2-\alpha_1)]=0. \qquad (33)$$

For the potentially stable arrangement, i.e. when $\alpha_2<\alpha_1$, the system is stable or unstable according as

$$k \lesseqgtr 1/\sqrt{2\,\nu'}\,, \qquad (34)$$

depending on the kinematic viscoelasticity ν'. Note that, in the absence of viscoelasticity ν', the system is always stable. Therefore the viscoelasticity has a destabilizing effect (for short waves) in this case. For the potentially unstable configuration, i.e. when $\alpha_2>\alpha_1$, we find that even if the condition

$$(\boldsymbol{k}\cdot\boldsymbol{V}_A)^2>g\,k\,(\alpha_2-\alpha_1) \qquad (35)$$

is satisfied, the medium can not be stabilized unless the condition (34) (i.e. $k<1/\sqrt{2\,\nu'}$) is also satisfied. In this case, (33) does not admit any change of sign, and it has not positive root. Therefore the system is stable. However, the condition for instability is given by

$$(\boldsymbol{k}\cdot\boldsymbol{V}_A)^2<g\,k\,(\alpha_2-\alpha_1)\,, \qquad (36)$$

which makes the absolute term of (33) negative, and therefore will possess one real positive root which destabilizes the system for Walters B′ elastico-viscous fluid.

Thus for the unstable case $\alpha_2>\alpha_1$, both the magnetic field and the kinematic viscoelasticity (where $2\,\nu'\,k^2<1$) have stabilizing effects, and the system is stable for all wavenumbers which satisfy the inequalities

$$\frac{1}{\sqrt{2\nu'}}>k>\frac{g\,(\alpha_2-\alpha_1)}{(V_1\,\cos\theta+V_2\,\sin\theta^2)}\,, \qquad (37)$$

otherwise the system will be unstable, where V_1 and V_2 are the Alfvén velocities in the x and y directions, respectively, and θ is the angle between $\boldsymbol{k}$ and H_x. Note that, if $\alpha_2>\alpha_1$ and $k>1/\sqrt{2\,\nu'}$, (33) will possess at least one real positive root which will destabilize the system for all wavenumbers irrespective of $(\boldsymbol{k}\cdot \boldsymbol{V}_A)^2 \gtrless g\,k\,(\alpha_2-\alpha_1)$. Also if $\alpha_2>\alpha_1$ and $(\boldsymbol{k}\cdot \boldsymbol{V}_A)^2<g\,k\,(\alpha_2-\alpha_1)$, the system will be unstable for all wavenumbers even in the presence of kinematic viscoelasticity ν'.

The stability criterion (37) is independent of the effects of viscosity. The magnetic field is found to stabilizes a certain wavenumbers range of the unstable configuration even in the presence of the effect of viscosity, while the kinematic viscoelasticity is found to have a stabilizing as well as destabilizing effect, depending on the chosen wavenumber values in this case. The critical wavenumbers k^*, given by (37), above (or below) which the system stabilizes, depends of the magnitudes V_1 and V_2 of the Alfvén velocities, the orientation θ of the magnetic field as well as the kinematic viscoelasticity ν'.

We now examine the behaviour of growth rates with respect to Alfvén velocities analytically. Since for $\alpha_2>\alpha_1$, (33) has one positive root let n_0 denote this positive root. Then

$$(1-2\,\nu' k^2)\,n_0^2+2\,\nu\, k^2\, n_0 + [(\boldsymbol{k}\cdot \boldsymbol{V}_A)^2-g\,k\,(\alpha_2-\alpha_1)]=0. \qquad (38)$$

It follows from (38) that

$$\frac{\mathrm{d}n_0}{\mathrm{d}V_j}=-\frac{k_j\,(\boldsymbol{k}\cdot\boldsymbol{V}_A)}{n_0\,(1-2\nu' k^2)+\nu\, k^2}, \quad j=1,2. \qquad (39)$$

It is evident from (39) that $\mathrm{d}n_0/\mathrm{d}V_j$ are positive or negative depending on whether the denominator $n_0(1-2\,\nu' k^1) + \nu\, k^2$ is negative or positive, respectively, i.e. for wavenumbers $k \gtrless n\sqrt{n_0/(2\,n_0\,\nu'-\nu)}$ which depend upon both the kinematic viscosity and viscoelasticity related by the relation $\nu<2\,n_0\,\nu'$. The growth rates, therefore, decrease as well as increase with increase of the Alfvén velocities. Therefore the Alfvén velocities have a dual role on the instability problem. Note also that (39) confirms the stability of the system in the presence of the magnetic field if the condition $k<1/\sqrt{2\,\nu'}$, as given before by (34), is satisfied.

4. Exponentially Varying Stratifications

Assume the stratification in fluid density, viscosity, viscoelasticity, and magnetic field to be of the form

$$(\varrho,\mu,\mu',H^2)=(\varrho_0,\mu_0,\mu_0',H_0^2)\exp(\beta z), \qquad (40)$$

where ϱ_0, μ_0, μ_0', H_0, and β are constants. Equation (40) implies that the kinematic viscosity $\nu\,(=\mu/\varrho=\mu_0/\varrho_0=\nu_0)$, kinematic viscoelasticity $\nu'=\mu'/\varrho'=\mu_0'/\varrho_0'=\nu_0'$), and Alfvén velocity $V_A\,(=\sqrt{\mu_e\,H^2/4\,\pi\,\varrho})=\sqrt{\mu_e\,H_0^2/4\,\pi\,\varrho_0})$ are constants everywhere.

Substituting (40) into (19) (after putting it in the suitable form, taking into account that the magnetic field is variable), and neglecting the effect of heterogeneity on the inertia, we obtain

$$[n^2+(\boldsymbol{k}\cdot \boldsymbol{V}_A)^2-n\,\beta^2(\nu_0-\nu_0'\,n)]\,(D^2-k^2)\,w - n\,(\nu_0-\nu_0'\,n)\,(D^2-k^2)^2\,w + [g\,k^2\,\beta-2\,n\,k^2\,\beta^2\,(\nu_0-\nu_0'\,n)]\,w=0. \qquad (41)$$

Considering the case of two free boundaries which, though a little artificial except for stellar atmospheres where it is the most appropriate [28], provides analytical solutions. The boundary conditions in this case are

$$w=D^2\,w=0 \quad \text{at} \quad z=0, \quad \text{and} \quad z=d. \qquad (42)$$

The appropriate solution of (41), satisfying the conditions (42), is

$$w=E\sin\left(\frac{a\,\pi\,z}{d}\right), \qquad (43)$$

where E is an arbitrary constant, and a is an integer.

Substituting (43) into (41), we obtain the dispersion relation

$$n^2\,\{L-\nu_0'\,[L^2+\beta^2\,(2\,k^2-L)]\} + n\,\nu_0\,[L^2+\beta^2\,(2\,k^2-L)]+L\,(\boldsymbol{k}\cdot \boldsymbol{V}_A)^2-g\,k^2\beta=0, \qquad (44)$$

where $L=k^2+(a\,\pi/d)^2$.

For $\beta<0$ (stable density stratification), and if the coefficient of n^2 is positive, i.e. if $\nu_0'<L/[L^2+\beta^2\,(2\,k^2-L)]$, (44) does not allow any positive root of n, and so the system is stable. For other values of the kinematic viscoelasticity the system is unstable. For $\beta>0$ (unstable density stratification), the system is stable or unstable according as

$$(\boldsymbol{k}\cdot \boldsymbol{V}_A)^2 \gtrless g\,k^2\,\beta/L. \qquad (45)$$

Note that, for the stability condition, all the coefficients of (44) should be real and positive. In this case we will have also the condition

$$\nu_0'<\frac{L}{[L^2+\beta^2\,(2\,k^2-L)]}, \qquad (46)$$

where $k\geq 1$ and $\beta\leq 3$; otherwise, the kinematic viscoelasticity ν' will have a destabilizing effect even in the presence of the magnetic field.

In the absence of the magnetic field, the system is unstable for $\beta>0$. However, the system can be completely stabilized by a large magnetic field, as can be seen from (44), if

$$(V_1 \cos\theta + V_2 \sin\theta) > g\,\beta/L \qquad (47)$$

and if conditioning (46) is still valid. The magnetic field, therefore, succeeds in stabilizing a wavenumber range

$$k^2 > \frac{g\,\beta}{(V_1 \cos\theta + V_2 \sin\theta)^2} - \left(\frac{a\,\pi}{d}\right)^2, \qquad (48)$$

which was unstable in the absence of the magnetic field. Note that the kinematic viscosity does not have any qualitative effect on the nature of stability or instability in this case.

However, if $\beta>0$, $(\boldsymbol{k}\cdot \boldsymbol{V}_A)^2 < g\,k^2\,\beta/L$, then (44) has one positive root. Let n_0 denotes the positive root of (44), then

$$n_0^2\,\{L - \nu_0'\,[L^2+\beta^2\,(2\,k^2-L)]\} + n_0\,\nu_0\,[L^2+\beta^2\,(2\,k^2-L)] + L(\boldsymbol{k}\cdot \boldsymbol{V}_A)^2 - g\,k^2\beta = 0. \qquad (49)$$

To find the role of the stratification parameter β on the growth rate of the unstable modes in this case, we examine the nature of $\mathrm{d}\,n_0/\mathrm{d}\,\beta$. Equation (48) yields

$$\frac{\mathrm{d}n_0}{\mathrm{d}\beta} = \frac{2\,n_0\,L + (\nu_0 - 2\,n_0\,\nu_0')\,[L^2+\beta^2(2\,k^2-L)]}{2\,n_0\,\beta\,(n_0\,\nu_0' - \nu_0)\,(2\,k^2-L) + g\,k^2} \qquad (50)$$

Now consider the inequalities

$$\{2\,n_0\,L + (\nu_0 - 2\,n_0\,\nu_0')\,[L^2+\beta^2\,(2\,k^2-L)]\} \gtrless 0 \qquad (51)$$

and

$$\{2\,n_0\,\beta\,(n_0\,\nu_0' - \nu_0)\,(2\,k^2-L) + g\,k^2\} \gtrless 0\,. \qquad (52)$$

If either both upper or both lower signs of (51) and (52) hold, then $\mathrm{d}n_0/\mathrm{d}\beta$ is positive. Thus we infer that the growth rate of unstable Rayleigh-Taylor modes increases with increasing the stratification parameter β, when the mentioned restrictions hold. Thus, the conditions (51) and (52) define the region where the parameter β has a destabilizing influence. But if the upper sign of the inequality (51), and the lower sign of (52), or vice versa, hold simultaneously, the growth rate turns out to be negative. This means, under these limitations, the parameter β can decreases the growth rate of the unstable Rayleigh-Taylor modes. We observe from (51) and (52) that the stabilizing or destabilizing influence of the stratification parameter β is independent of the magnetic field, and dependent both the kinematic viscosity and viscoelasticity as well as the parameter β. Finally, note that in the absence of stratification parameter β it follows from (49) that the magnetic field has stabilizing as well as destabilizing effects if the conditions $\nu_0'\,L \gtrless 1$ are respectively satisfied. Also, in the absence of both stratification and viscoelasticity, i.e. when $\beta=0$ and $\nu_0'=0$, (49) showed that the applied magnetic field has always a stabilizing effect [1].

In conclusion, this paper describes the effect of a horizontal magnetic field and a non-Newtonian stress tensor as described by the Walters B′ model, on the viscous Rayleigh-Taylor instability of a two-layer arrangement of second order fluids. It is found that there are two kinds of instabilities: The first one is the well-known instability of the second order fluids which is an artifact of the model, while the second one is due to the instability from density stratification, which can be stabilized for short waves by the magnetic field in both kinds. In the first kind of instability, if $\alpha_2<\alpha_1$, then the system will be stable or unstable according as $k \gtrless 1/\sqrt{2\,\nu'}$, where the kinematic viscoelasticity is found to have a destabilizing effect (for short waves). If $\alpha_2>\alpha_1$, then the system is stable only if the conditions $(\boldsymbol{k}\cdot \boldsymbol{V}_A)^2 > g\,k\,(\alpha_2-\alpha_1)$ and $k<1/\sqrt{2\,\nu'}$ hold simultaneously; otherwise the system will be unstable. Instability occurs also if $(\boldsymbol{k}\cdot \boldsymbol{V}_A)^2 > g\,k\,(\alpha_2-\alpha_1)$, irrespective of the fluid being viscoelastic or not. In the second kind of instability, if $\beta<0$, then the system will be stable or unstable according to $\nu_0' \lesseqgtr L/[L^2+\beta^2\,(2\,k^2-L)]$, where $k\geq 1$ and $-3<\beta<0$. If $\beta>0$, then the system is stable if the conditions $(\boldsymbol{k}\cdot \boldsymbol{V}_A)^2 > g\,k^2\,\beta/L$, and $\nu_0'<L/\,[L^2+\beta^2\,(2\,k^2-L)]$ hold simultaneously, where $k\geq 1$ and $0<\beta<3$. It is also found that viscoelasticity has a stabilizing effect (for long waves) as well as a destabilizing effect (for short waves) on the considered system where $|\beta|\leq 3$. Viscosity is found to have no effect on the stability boundary (see (37), (46), and (47)), and merely changes the growth rates of the other physical parameters such as the Alfvén velocities (in the first kind), or the stratification parameter (in the second kind) under certain conditions, as shown above.

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